You can uniquely recover the nomographic scales from the isopleth lines

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Abstract

A nomogram for $F(u,v,w) = 0$ consists of three curves in the plane $\gamma_i : \mathbb{R} \to \mathbb{R}^2$, parameterized by u, v, and w respectively. Specifying a value of u and v, you can draw the line through $\gamma_1(u)$ and $\gamma_2(v)$, find where the line intersects γ_3, and read off the solution $w = \widehat{w}(u,v)$ which satisfies the equation $F(u,v,w) = 0$.

The line has a particular slope $A(u,v)$ and y-intercept $B(u,v)$. Because nomograms are well-behaved, the correspondence $\langle u,v\rangle \longleftrightarrow \langle A,B\rangle$ is actually a smooth invertible *change in coordinates*. And if you happen to know the functions $A(u,v)$ and $B(u,v)$, you can uniquely recover all three curves γ_i. The formula, as I show here, is:

$$\gamma_i \circ \chi_i(u,v) \equiv \left\langle -\frac{J(B,\chi_i)}{J(A,\chi_i)}, \quad B - A \cdot \frac{J(B,\chi_i)}{J(A,\chi_i)} \right\rangle.$$

Here, $J(p,q)$ denotes the Jacobian $\frac{\partial p}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial p}{\partial y}\frac{\partial q}{\partial x}$. And we've defined the 'variable-solving' functions: $\chi_1(u,v) \equiv u$, $\chi_2(u,v) \equiv v$, and $\chi_3(u,v) \equiv \widehat{w}(u,v)$—each of them takes u and v and solves for one of u, v, or w, respectively.

A geometric curiosity is that the right hand side of this equation bears a family resemblance to the equation for finding the x-intercept of the line $y = ax + b$ (as $x_0 = -b/a$), and then plugging that intercept back into the equation for line (as $y = b + ax_0$).

Preliminaries A nomogram for $F(u,v,w) = 0$ consists of three curves in the plane $\gamma_i : \mathbb{R} \to \mathbb{R}^2$, parameterized by u, v, and w respectively.

Without loss of generality, we'll pick w to be the designated distinguished variable among the three, and write $\widehat{w}(u,v)$ as the function that is implicitly defined by solving $F(u,v,w) = 0$ for the variable w.

Nomographic scales never intersect The first well-behaved condition is that nomographic scales never intersect. In other words, given any pair of values u and v, we can plot the corresponding points on their curves $\gamma_1(u)$ and $\gamma_2(v)$ and draw the unique line through them.

Note that we can specify that line using the pair of values $\langle u, v\rangle$ or by specifying the slope $A(u,v)$ and intercept $B(u,v)$ of the line. (I call $A(u,v)$ the *slope field* and $B(u,v)$ the *intercept field*. The line drawn through the nomographic scales is sometimes called an *isopleth* line.)

The second well-behaved condition is that, conversely, if we draw any line through the plane, it'll intersect each nomographic scale in (at most) one point. In the next section, I'll show how if you know $A(u,v)$ and $B(u,v)$, you can solve uniquely for the curves γ_1 and γ_2 (and in fact γ_3).

The correspondence $\langle u, v\rangle \leftrightarrow \langle A, B\rangle$ is a coordinate transform The curves of a nomogram are well behaved. From this fact that we can uniquely specify pairs of points on our two curves either using the parameters $\langle u, v\rangle$ or by using the slope-intercept $\langle A, B\rangle$ means that the correspondence

$$\langle u, v\rangle \leftrightarrow \langle A, B\rangle$$

is actually a *coordinate transform* — that is to say, a smooth invertible transformation where every pair has a unique counterpart. The formal way of saying this is that the Jacobian $J(A,B) \equiv \frac{\partial A}{\partial u}\frac{\partial B}{\partial v} - \frac{\partial A}{\partial v}\frac{\partial B}{\partial u}$ is nonzero everywhere.

1 Solving for the curve γ_i given the isopleth line

Suppose you know the slope field $A(u,v)$ and intercept field $B(u,v)$ and you want to recover the parametric curve $\gamma_1(u)$ from it.

This is actually fairly easy to do: fix a value of u and pick two values $v_1 \neq v_2$. Then we'll have the two lines corresponding to $\langle A(u,v_1), B(u,v_1)\rangle$ and $\langle A(u,v_2), B(u,v_2)\rangle$, whose unique point of intersection will tell us where $\gamma_1(u)$ is. [1]

We will assume v_1 and v_2 are sufficiently close to each other, both for smoothness reasons—if they're sufficiently close points on a well-behaved

[1] Detail checking: You might worry about the pathological case where these two lines don't have a unique point of intersection—if the lines are parallel or identical. But in fact, this pathological case can't happen as long as v_1 and v_2 are sufficiently close. This is because nomogrammable functions are 'well-behaved' — roughly speaking, if you keep u fixed and perturb v a little bit, the isopleth line must necessarily change to pass through the new point on the v scale. And yet the line still passes through the same 'pivot' point on the u scale, so the change must have altered the slope of the isopleth line—the new line is not parallel to the old line.

nomogramic scale, the two lines $\langle A_1, B_1 \rangle$ and $\langle A_2, B_2 \rangle$ cannot be parallel—and because in the end we'll take the limit as $v_2 \to v_1$ so that we can express our answer neatly in terms of partial derivatives of A and B.

As we will show, the coordinates of the point $\gamma_1(u) = \langle f_1(u), g_1(u) \rangle$ can be recovered as[2]:

$$\begin{cases} f_1(u) = -\dfrac{\partial_2 B(u,v)}{\partial_2 A(u,v)} \\ g_1(u) = B(u,v) - A(u,v) \cdot \dfrac{\partial_2 B(u,v)}{\partial_2 A(u,v)} \end{cases} \tag{1}$$

Here's a nice geometric observation in passing: these equations bear a sort of family resemblance to the formula for finding the x-intercept of the line $y = ax + b$—namely $x_0 \equiv -b/a$— and for plugging that value back in to the original equation: $y = ax_0 + b$.

Proof of the isopleth recovery formula for $\gamma_1(u)$. Fix a value u and fix two values we'll call v and $v + dv$ (anticipating the final step where we'll take a limit $dv \to 0$). The pairs $\langle u, v \rangle$ and $\langle u, v + dv \rangle$ define two lines $\langle A(u,v), B(u,v) \rangle$ and $\langle A(u,\, v + dv), B(u,\, v + dv) \rangle$. At their intersection is the point $\gamma_1(u)$ which we're trying to recover in terms of A and B.

The $\langle x, y \rangle$ coordinates of that point satisfy two linear equations, namely the equations that say that the point lies on each of our lines:

$$\begin{cases} y = & A(u,\, v)\, x + B(u,v) \\ y = & A(u,\, v + dv)\, x + B(u,\, v + dv) \end{cases}$$

In matrix form, this is the same as:

$$\begin{bmatrix} -A(u,v) & 1 \\ -A(u,\, v + dv) & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} B(u,v) \\ B(u, v + dv) \end{bmatrix}$$

To solve this system of equations, we'll have to invert that coefficient matrix on the left. Its inverse is:

$$\frac{1}{A(u, v + dv) - A(u,v)} \begin{bmatrix} 1 & -1 \\ A(u, v + dv) & -A(u,v) \end{bmatrix},$$

which means our solution is given by:

[2]Note that I use Spivak notation such as $\partial_1 p(x,y)$ and $\partial_2 p(x,y)$ instead of the usual $\partial p / \partial x$ and $\partial p / \partial y$, respectively. The notation $\partial_i p$ means "the derivative of p with respect to its ith argument."

$$\begin{aligned}
\begin{bmatrix} x \\ y \end{bmatrix} &= \frac{1}{A(u,v+dv)-A(u,v)} \begin{bmatrix} 1 & -1 \\ A(u,v+dv) & -A(u,v) \end{bmatrix} \begin{bmatrix} B(u,v) \\ B(u,v+dv) \end{bmatrix} \\
&= \frac{1}{A(u,v+dv)-A(u,v)} \begin{bmatrix} B(u,v)-B(u,v+dv) \\ A(u,v+dv)B(u,v)-A(u,v)B(u,v+dv) \end{bmatrix}
\end{aligned}$$

In the second row, we'll cleverly add and subtract an extra $A(u,v)B(u,v)$ term to make it look more like some derivatives:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{A(u,v+dv)-A(u,v)} \begin{bmatrix} B(u,v)-B(u,v+dv) \\ [A(u,v+dv)-A(u,v)]B(u,v)-A(u,v)[B(u,v+dv)-B(u,v)] \end{bmatrix}$$

Multiplying and dividing by dv, then taking the limit as $dv \to 0$:

$$\begin{aligned}
\begin{bmatrix} x \\ y \end{bmatrix} &= \frac{dv}{A(u,v+dv)-A(u,v)} \begin{bmatrix} \dfrac{B(u,v)-B(u,v+dv)}{dv} \\ \dfrac{A(u,v+dv)-A(u,v)}{dv} B(u,v) - A(u,v) \dfrac{B(u,v+dv)-B(u,v)}{dv} \end{bmatrix} \\
&\overset{dv \to 0}{=} \frac{1}{\partial_2 A(u,v)} \begin{bmatrix} -\partial_2 B(u,v) \\ \partial_2 A(u,v) \cdot B(u,v) - A(u,v) \cdot \partial_2 A(u,v) \end{bmatrix}
\end{aligned}$$

In short, after some slight algebraic simplification, we've solved for the x and y coordinates of our point $\gamma_1(u)$:

$$\begin{cases} x = -\dfrac{\partial_2 B(u,v)}{\partial_2 A(u,v)} \\ y = B(u,v) - A(u,v) \cdot \dfrac{\partial_2 B(u,v)}{\partial_2 A(u,v)} \end{cases}$$

2 A unified formula for recovering the curves from the isopleth

We've seen how to recover $\gamma_1(u)$ from the isopleth's slope field $A(u,v)$ and intercept field $B(u,v)$, namely

$$\gamma_1(u) = \left\langle -\frac{\partial_2 B}{\partial_2 A}, \quad B - A \cdot \frac{\partial_2 B}{\partial_2 A} \right\rangle$$

An exactly analogous argument lets you recover $\gamma_2(v)$ as

$$\gamma_2(v) = \left\langle -\frac{\partial_1 B}{\partial_1 A}, \quad B - A \cdot \frac{\partial_1 B}{\partial_1 A} \right\rangle$$

It is also possible to solve for our third curve $\gamma_3(w)$, though the situation is a little different because we are dealing with our distinguished variable w. Note, for example, that $A(u,v)$ and $B(u,v)$ are functions of u and v only, and so we cannot really put them together to obtain a function of w—instead, the appropriate concept here is to solve for $\gamma_3(\widehat{w}) = \gamma_3(\widehat{w}(u,v))$. With basically the same trick as before—we pick two nearby points on the u curve and two nearby points on the v curve such that $\langle u_1, v_1\rangle$ and $\langle u_2, v_2\rangle$ have the same $\widehat{w}(u,v)$ value[3], then find the intersection of the corresponding lines—we find, after much algebra crunching, that:

$$\gamma_3 \circ \widehat{w} = \left\langle -\frac{J(B,\widehat{w})}{J(A,\widehat{w})}, \quad B - A \cdot \frac{J(B,\widehat{w})}{J(A,\widehat{w})} \right\rangle,$$

where J represents the Jacobian. In fact, all three formulas can be put into common form. If we know the values of u and v, we know we can solve for $w = \widehat{w}(u,v)$. We can also trivially "solve" for u or v. Let us define the three variable-solving functions:

$$\begin{aligned}
\chi_1(u,v) &= u \\
\chi_2(u,v) &= v \\
\chi_3(u,v) &= \widehat{w}(u,v)
\end{aligned}$$

Then we can compactly represent all three of our formulas as:

$$\gamma_i \circ \chi_i(u,v) = \left\langle -\frac{J(B,\chi_i)}{J(A,\chi_i)}, \quad B - A \cdot \frac{J(B,\chi_i)}{J(A,\chi_i)} \right\rangle$$

Because in fact $J(\cdot,\chi_1) = -\partial_2$ and $J(\cdot,\chi_2) = \partial_1$ are just partial derivative operators.

3 A corollary about A and B

In more ordinary language, we might have written our first two equations as:

$$\gamma_1(u) = \left\langle -\frac{\partial_2 B}{\partial_2 A}, \quad B - A \cdot \frac{\partial_2 B}{\partial_2 A} \right\rangle$$

[3] You can do this by finding your first pair $\langle u, v\rangle$. Then use the gradient of $\widehat{w}(u,v)$ to find a nearby pair $\langle u', v'\rangle$ that have the same $\widehat{w}$ value. This is always possible because if you move a little bit on the u scale, you can always move a compensatory amount on the v scale so that the isopleth still passes through the same point on the w scale as before.

$$\gamma_2(v) = \left\langle -\frac{\partial_1 B}{\partial_1 A}, \quad B - A \cdot \frac{\partial_1 B}{\partial_1 A} \right\rangle$$

then the fact that the left sides are functions of one variable means that if we take the derivative with respect to the other variable, we'll get an expression equal to zero.

What we will do now is essentially that same operation in a more high-brow notation. The Jacobian J has the property that for any smooth $p : \mathbb{R}^2 \to \mathbb{R}^1$ and $q : \mathbb{R}^2 \to \mathbb{R}^2$, the Jacobian of $p \circ q$ and q vanishes everywhere:

$$J(p \circ q, q) = 0.$$

You can prove this using the chain rule. Qualitatively, the Jacobian measures the degree to which two functions are independent, and so the Jacobian of q and a function of q is zero everywhere because they're completely dependent.

It will also be useful to know the product rule and chain rule for Jacobians (properties which are easy to derive by working through the definition of the Jacobian).

$$J(p/q,\, r) = \frac{qJ(p,r) - pJ(q,r)}{q^2}$$

$$J(p \cdot q,\, r) = pJ(q,r) + qJ(p,r)$$

$$J(p \circ q,\, r) = (Dp \circ q) \cdot J(q,r)$$

With those tools in hand, let us return to our isopleth equation:

$$\gamma_i \circ \chi_i(u,v) = \left\langle -\frac{J(B,\chi_i)}{J(A,\chi_i)}, \quad B - A \cdot \frac{J(B,\chi_i)}{J(A,\chi_i)} \right\rangle$$

If we apply the Jacobian operator $J(-,\chi_i)$ to both sides, we get zero on the left, as $J(\gamma_i \circ \chi_i,\, \chi_i) = 0$. On the right, we apply our Jacobian rules and find:

$$0 = -\frac{J(A,\chi_i) \cdot J[J(B,\chi_i),\chi_i] \;-\; J(B,\chi_i) \cdot J[J(A,\chi_i),\chi_i]}{J(A,\chi_i)^2}$$

(Technically, our original expression was a vector with two components, so we should have two components here too—but as it turns out, when we calculate the second component we get $A(u,v)$ times the first component, and since both components are equal to zero, the content is redundant.)

Cryptic Gronwall footnote

$$\mathcal{A} \equiv +\frac{J\left(-\frac{J(B,v)}{J(A,v)}, v\right) \cdot J(A,v)^2}{J(A,B)}$$
$$\mathcal{B} \equiv -\frac{J\left(-\frac{J(B,u)}{J(A,u)}, u\right) \cdot J(A,u)^2}{J(A,B)}$$
$$\mathcal{C} \equiv +\frac{J(J(A,B),v) - 3 \cdot J(-J(B,v)/J(A,v),u) \cdot J(A,v)^2}{J(A,B)}$$
$$\mathcal{D} \equiv -\frac{J(J(A,B),u) - 3 \cdot J(-J(B,u)/J(A,u),v) \cdot J(A,u)^2}{J(A,B)}$$

$$\mathcal{C} \equiv +\frac{J(J(A,B),v) - 3 \cdot f_1'(u) \cdot J(A,v)^2}{J(A,B)}$$
$$\mathcal{D} \equiv -\frac{J(J(A,B),u) - 3 \cdot f_2'(v) \cdot J(A,u)^2}{J(A,B)}$$