You can recover the triple of nomographic curves uniquely from their isopleth lines

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This brief note contains a result about curves, lines, and slopes which I have derived. The result is important to nomography, but doesn't require knowledge of nomography to understand.

First, suppose you have two curves in space: $\gamma_1, \gamma_2 : \mathbb{R} \to \mathbb{R}^2$. Each curve maps a parameter (you might think of it as "time") to a particular position in space. Now, I can specify any two values—call them u and v—and plot the points $\gamma_1(u)$ and $\gamma_2(v)$. ("Locate the first curve at time u, and locate the second curve at time v".) Assuming the curves are well-behaved (e.g. they don't intersect), I can then draw the line in the plane between the points $\gamma_1(u)$ and $\gamma_2(v)$. And I can draw such a line for any two values u and v. Hence I can treat that line as a function of u and v, letting $A(u, v)$ be the slope of the line and letting $B(u, v)$ be the y-intercept.

The first question is: if you have forgotten the curves $\gamma_1(u)$ and $\gamma_2(v)$, but you have all the information about such lines, namely slope $A(u, v)$ and y-intercept $B(u, v)$, can you uniquely recover the curves γ_1 and γ_2? As I will show, you can indeed uniquely recover the curves.

Now for the second question. Suppose in addition to the curves $\gamma_1(u)$ and $\gamma_2(v)$, now you have a third curve $\gamma_3(w)$ with an interesting property. As before, for any u and v you can draw the line joining $\gamma_1(u)$ and $\gamma_2(v)$. The curve $\gamma_3(w)$ has the property that time you draw a line connecting $\gamma_1(u)$ to $\gamma_2(v)$, it will intersect the curve γ_3 somewhere; we can refer to the value of w at which this occurs as $\hat{w}(u, v)$. Then our assumption is: for any u and v, the point $\gamma_3(\hat{w}(u, v))$ always lies on the line joining $\gamma_1(u)$ and $\gamma_2(v)$. The question is if all we know is $A(u, v)$ and $B(u, v)$ and $\hat{w}(u, v)$, can we now uniquely recover this third curve $\gamma_3(w)$? Again, I will show that you can indeed uniquely recover the third curve.

We have the following theorem:

1 Theorem Suppose you are given the slope field $A(u, v)$ and the intercept field $B(u, v)$ for a pair of curves $\gamma_1(u)$ and $\gamma_2(v)$. (That is, for each pair of points u and v, $A(u, v)$ represents the slope and $B(u, v)$ represents the y-intercept of the line between $\gamma_1(u)$ and $\gamma_2(v)$.) Then, except in extreme edge cases, you can *uniquely* recover γ_1 and γ_2 given only $A(u, v)$ and $B(u, v)$.

Proof. The proof is by a simple geometric construction: for each value u, we pick two different values v_0 and v_1. We know the slope and intercept of the line joining $\gamma_1(u)$ and $\gamma_2(v_0)$ and the line joining $\gamma_1(u)$ and $\gamma_2(v_1)$. Hence we plot those two lines; if they intersect at all, it will be at a unique point which yields $\gamma_1(u)$. This is enough to establish the proof.

This argument assumes that we can find v_0 and v_1 such that the resulting lines actually intersect at one point (i.e. are not parallel). But in fact, these curves are nomographic and so they must be well-behaved. In particular, they ought to have the following property: "changing one of the values u v must necessarily change the line between them". We describe this property formally by saying that the Jacobian $\partial(A, B) \equiv \frac{\partial A}{\partial u}\frac{\partial B}{\partial v} - \frac{\partial A}{\partial v}\frac{\partial B}{\partial u}$ is nonzero everywhere. This property is enough to guarantee that for every u, v_0 pair, there exists a v_1 near to v_0 which yields a different line.

□

We have just proved that we can recover the two curves $\gamma_1(u)$ and $\gamma_2(v)$ uniquely from their slope and intercept fields. Next, I'll show how we can also uniquely recover the third curve, $\gamma_3(w)$, given only the slope $A(u, v)$ and intercept $B(u, v)$ fields for the other two curves. Note that if $F(u, v, w) = 0$ is nomogramable, then presumably it has the following property: changing any one of the parameters by itself changes the value of F. (This need not necessarily be true, e.g. when $F(u, v, w) = uv - w$ at $u = 0$, but it should often be true.) In this case, the partial derivatives of F are nonzero everywhere, and the implicit function theorem says that the relation $F(u, v, w) = 0$ implicitly defines w as a function of u and v—call it $\hat{w}(u, v)$—and that this function is (locally) unique!

Using $\hat{w}$, we have the following result:

2 Theorem Suppose you are given the slope field $A(u, v)$ and the intercept field $B(u, v)$ for a pair of curves $\gamma_1(u)$ and $\gamma_2(v)$. You are also given the function $\hat{w}(u, v)$ which defines w as a function of u and v. Then, except in extreme edge cases, you can *uniquely* recover $\gamma_3 \circ \hat{w}$ (this is γ_3 expressed as a function of u and v.)

Specifically, if we use the notation $\gamma_i \equiv \langle f_i, g_i \rangle$ so we can refer to the individual coordinates of the curves, we have that $\gamma_3 \circ \hat{w}$ is uniquely defined by the equations:

$$(f_3 \circ \hat{w})(u, v) = -\frac{\partial(B, \hat{w})}{\partial(A, \hat{w})}$$

$$(g_3 \circ \hat{w})(u, v) = B - A\frac{\partial(B, \hat{w})}{\partial(A, \hat{w})}$$

Here, I am using the shorthand notation $\partial(f, g)$ for the Jacobian determinant $\frac{\partial f}{\partial u}\frac{\partial g}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial g}{\partial u}$.

Proof. In brief, if you can find two pairs of values $\langle u, v \rangle$ and $\langle u', v' \rangle$ which yield the same value w, you can plot the two lines and find $\gamma_3 \circ \hat{w}$ as their unique point of intersection. In this proof, I use the derivatives of $\hat{w}$ to find such pairs of points for any

w. Then I use the standard equation for line-line intersection to compute that intersection point. Through some clever algebra, I manipulate the resulting expressions to express them solely in terms of A, B, and $\hat{w}$.

Let's fix a particular value $\langle u, v \rangle$ and corresponding $w \equiv \hat{w}(u, v)$. We must find another value $\langle u', v' \rangle$ such that $\hat{w}(u', v') = w = \hat{w}(u, v)$.

It turns out that if the partial derivatives of F are nonzero everywhere, then so are the partial derivatives of $\hat{w}$. This implies that the *gradient* of $\hat{w}(u, v)$ is nonzero everywhere. But if you travel in $u - v$ space perpendicular to the gradient, then $\hat{w}$ will remain locally constant. Hence, if you have a value $\langle u, v \rangle$, you can find another nearby value $\langle u', v' \rangle$ which yields the same value of w, by traveling a short distance away from $\langle u, v \rangle$ in a direction perpendicular to the gradient.

For concreteness, one such direction perpendicular to the gradient of $\hat{w}$ is $\vec{n} \equiv \langle \frac{\partial \hat{w}}{\partial v}, -\frac{\partial \hat{w}}{\partial u} \rangle$. This choice of direction defines our second pair as:

$$
\begin{aligned}
u' &= u + \epsilon \frac{\partial \hat{w}}{\partial v} \\
v' &= v - \epsilon \frac{\partial \hat{w}}{\partial u}
\end{aligned}
$$

where we require ϵ, the step size, to be very small. (We will be taking the limit $\epsilon \to 0$ later.)

Now that we have $\langle u, v \rangle$ and $\langle u', v' \rangle$, we have two lines: the one defined by $A(u, v)$ and $B(u, v)$, and the one defined by $A(u', v')$ and $B(u', v')$. Of course the intersection of those lines will be at $\gamma_3(w)$, i.e. at the point $\langle f_3 \circ \hat{w}, g_3 \circ \hat{w} \rangle$, which is what we want to find. Evidently, if the point of intersection is unique, then that particular point of $\gamma_3(w)$ is uniquely defined by A, B, and $\hat{w}$.

There is a well-known expression for finding the intersection of two lines using determinants. It's rather complicated, so I won't reproduce it here. Into that expression, we plug in our four points $\langle f_1(u), g_1(u) \rangle$, $\langle f_2(v), g_2(v) \rangle$, $\langle f_1(u'), g_1(u') \rangle$, $\langle f_2(v'), g_2(v') \rangle$. This yields a complicated mess. To simplify, we define new functions

$$
\begin{aligned}
P(u, v) &\equiv f_1(u) - f_2(v) \\
Q(u, v) &\equiv f_1(u) g_2(v) - g_1(u) f_2(v) \\
R(u, v) &\equiv g_1(u) - g_2(v)
\end{aligned}
$$

With those substitutions, the equation for the intersection of our two lines becomes much more manageable:

$$
\begin{aligned}
(f_3 \circ \hat{w}) &= \lim_{\epsilon \to 0} \frac{Q(u, v)P(u', v') - P(u, v)Q(u', v')}{P(u, v)R(u', v') - R(u, v)P(u', v')} \\
(g_3 \circ \hat{w}) &= \lim_{\epsilon \to 0} \frac{Q(u, v)R(u', v') - R(u, v)Q(u', v')}{P(u, v)R(u', v') - R(u, v)P(u', v')}
\end{aligned}
$$

Our next plan is to make the numerators and denominators look more like difference quotients, i.e. roughly like $\frac{h(x+\epsilon)-h(x)}{\epsilon}$, so that when we take the limit, we will get a bunch of derivatives.

To do so, we simply add and subtract some extra stuff from each numerator and denominator:

$$(f_3 \circ \hat{w}) = \lim_{\epsilon \to 0} \frac{Q(u,v)P(u',v') + [Q(u,v)P(u,v) - Q(u,v)P(u,v)] - P(u,v)Q(u',v')}{P(u,v)R(u',v') + [P(u,v)R(u,v) - P(u,v)R(u,v)] - R(u,v)P(u',v')}$$
$$(g_3 \circ \hat{w}) = \lim_{\epsilon \to 0} \frac{Q(u,v)R(u',v') + [Q(u,v)R(u,v) - Q(u,v)R(u,v)] - R(u,v)Q(u',v')}{P(u,v)R(u',v') + [P(u,v)R(u,v) - P(u,v)R(u,v)] - R(u,v)P(u',v')}$$

We consolidate terms neatly, yielding:

$$(f_3 \circ \hat{w}) = -\lim_{\epsilon \to 0} \frac{P(u,v)\,[Q(u',v') - Q(u,v)] - Q(u,v)\,[P(u',v') - P(u,v)]}{P(u,v)\,[R(u',v') - R(u,v)] - R(u,v)\,[P(u',v') - P(u,v)]}$$
$$(g_3 \circ \hat{w}) = \lim_{\epsilon \to 0} \frac{Q(u,v)\,[R(u',v') - R(u,v)] - R(u,v)\,[Q(u',v') - Q(u,v)]}{P(u,v)\,[R(u',v') - R(u,v)] - R(u,v)\,[P(u',v') - P(u,v)]}$$

Then we divide numerator and denominator by ϵ:

$$(f_3 \circ \hat{w}) = -\lim_{\epsilon \to 0} \frac{P(u,v)\frac{Q(u',v')-Q(u,v)}{\epsilon} - Q(u,v)\frac{P(u',v')-P(u,v)}{\epsilon}}{P(u,v)\frac{R(u',v')-R(u,v)}{\epsilon} - R(u,v)\frac{P(u',v')-P(u,v)}{\epsilon}}$$
$$(g_3 \circ \hat{w}) = \lim_{\epsilon \to 0} \frac{Q(u,v)\frac{R(u',v')-R(u,v)}{\epsilon} - R(u,v)\frac{Q(u',v')-Q(u,v)}{\epsilon}}{P(u,v)\frac{R(u',v')-R(u,v)}{\epsilon} - R(u,v)\frac{P(u',v')-P(u,v)}{\epsilon}}$$

We are now ready to evaluate the limit as $\epsilon \to 0$; when we do, we will get several partial derivatives.

Side remark: I should like to point out that in general, if $h(u,v)$ is any two-variable function and $\hat{n}$ is any two-dimensional unit vector, then the partial derivative of h in the direction $\hat{n}$ is given by the limit

$$\partial_{\hat{n}} h(\vec{u}) = \lim_{\epsilon \to 0} \frac{h(\vec{u} + \epsilon\hat{n}) - h(\vec{u})}{\epsilon}$$

which is exactly the sort of thing we have going on in our expressions for $(f_3 \circ \hat{w})$ and $(g_3 \circ \hat{w})$ because we defined $\langle u', v' \rangle = \langle u, v \rangle + \epsilon\vec{n}$. Another way of writing the partial derivative of h in the direction $\hat{n}$ is as a dot product of $\hat{n}$ with the gradient of h:

$$\partial_{\hat{n}} h(\vec{u}) = \lim_{\epsilon \to 0} \frac{h(\vec{u} + \epsilon\hat{n}) - h(\vec{u})}{\epsilon} = \nabla h \cdot \hat{n} = \left\langle \frac{\partial h}{\partial u}, \frac{\partial h}{\partial v} \right\rangle \cdot \langle \hat{n}_x, \hat{n}_y \rangle\,.$$

When we evaluate the limit, we get[1]:

$$
\begin{aligned}
(f_3 \circ \hat{w}) &= -\frac{P(u,v)(\nabla Q \cdot \vec{n}) - Q(u,v)(\nabla P \cdot n)}{P(u,v)(\nabla R \cdot \vec{n}) - R(u,v)(\nabla P \cdot \vec{n})} \\
(g_3 \circ \hat{w}) &= \frac{Q(u,v)(\nabla R \cdot \vec{n}) - R(u,v)(\nabla Q \cdot n)}{P(u,v)(\nabla R \cdot \vec{n}) - R(u,v)(\nabla P \cdot \vec{n})}
\end{aligned}
$$

Recall that we defined $\vec{n} \equiv \langle \frac{\partial \hat{w}}{\partial v}, -\frac{\partial \hat{w}}{\partial u} \rangle$. Hence when we evaluate a term like $\nabla Q \cdot \vec{n}$, we get

$$
\nabla Q \cdot \vec{n} = \left\langle \frac{\partial Q}{\partial u}, \frac{\partial Q}{\partial v} \right\rangle \cdot \left\langle \frac{\partial \hat{w}}{\partial v}, -\frac{\partial \hat{w}}{\partial u} \right\rangle = \frac{\partial Q}{\partial u}\frac{\partial \hat{w}}{\partial v} - \frac{\partial Q}{\partial v}\frac{\partial \hat{w}}{\partial u}.
$$

The expression on the far right is called the Jacobian of Q and $\hat{w}$, and I'll write it in shorthand as $\partial(Q, \hat{w})$. The same reasoning we used to discover that $\nabla Q \cdot \vec{n} = \partial(Q, \hat{w})$ applies to the other functions P and R. Hence we have:

$$
\begin{aligned}
(f_3 \circ \hat{w}) &= -\frac{P(u,v)\,\partial(Q,\hat{w}) - Q(u,v)\,\partial(P,\hat{w})}{P(u,v)\,\partial(R,\hat{w}) - R(u,v)\,\partial(P,\hat{w})} \\
(g_3 \circ \hat{w}) &= -\frac{Q(u,v)\,\partial(R,\hat{w}) - R(u,v)\,\partial(Q,\hat{w})}{P(u,v)\,\partial(R,\hat{w}) - R(u,v)\,\partial(P,\hat{w})}
\end{aligned}
$$

I note that each numerator and denominator looks sort of like the quotient rule for derivatives. If you try to derive a quotient rule for *Jacobians*, you get the following:

$$
\partial(f/g,\ h) = \frac{g\,\partial(f,h) - f\,\partial(g,h)}{g^2}
$$

or, how I like to put it,

$$
g\,\partial(f,h) - f\,\partial(g,h) = g^2\,\partial(f/g,h).
$$

Applying this quotient rule to the numerators and to the denominators, we obtain:

$$
\begin{aligned}
(f_3 \circ \hat{w}) &= -\frac{\cancel{P^2}\ \partial(Q/P,\hat{w})}{\cancel{P^2}\,\partial(R/P,\hat{w})} \\
(g_3 \circ \hat{w}) &= \frac{Q^2\ \partial(R/Q,\hat{w})}{P^2\,\partial(R/P,\hat{w})}
\end{aligned}
$$

[1] It's not a problem that our vector $\vec{n} \equiv \langle \frac{\partial \hat{w}}{\partial v}, -\frac{\partial \hat{w}}{\partial u} \rangle$ is not normalized: we can factor out the length in the numerator and in the denominator, and so consequently it cancels out.

Now comes the truly exciting part. Observe that by definition,

$$A(u,v) = \frac{R(u,v)}{P(u,v)}$$
$$B(u,v) = \frac{Q(u,v)}{P(u,v)}$$

Hence $\frac{R}{P} = A$, $\frac{Q}{P} = B$, and $\frac{R}{Q} = \frac{R}{P}\frac{P}{Q} = \frac{A}{B}$. With these substitutions, we find that:

$$(f_3 \circ \hat{w}) = -\frac{\partial(B,\hat{w})}{\partial(A,\hat{w})}$$
$$(g_3 \circ \hat{w}) = B^2 \frac{\partial(A/B,\hat{w})}{\partial(A,\hat{w})}$$

We can simplify the expression for $(g_3 \circ \hat{w})$ one final time by using another application of the quotient rule for Jacobians. Specifically, $B^2\,\partial(A/B,\hat{w}) = B\,\partial(A,\hat{w})) - A\,\partial(B,\hat{w})$.

Hence we arrive at our final result:

$$(f_3 \circ \hat{w}) = -\frac{\partial(B,\hat{w})}{\partial(A,\hat{w})}$$
$$(g_3 \circ \hat{w}) = \frac{B\,\partial(A,\hat{w}) - A\,\partial(B,\hat{w})}{\partial(A,\hat{w})} = B - A\frac{\partial(B,\hat{w})}{\partial(A,\hat{w})}$$

□

This equation is neat because the equation for $(f_3 \circ \hat{w})$, the x-coordinate of γ_3, looks analogous to the equation for finding the x-intercept of a line $y = Ax + B$. Similarly, if you substitute the equation for f_3 into the equation for $(g_3\hat{w})$, you get the actual equation of the line $(g_3 \circ \hat{w}) = A(f_3 \circ \hat{w}) + B$.