Kellogg's two criteria for nomographability *plus*
Warmus's nomographic procedure

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03/Jan/2020

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1 Kellogg's two nomographability criteria

Fix $F(x, y, z)$, and pick one of the variables, say z.

Kellogg's first necessary criterion. If F can be represented directly* as a nomogram, then the following matrix must have less than full rank:

$$\begin{bmatrix} F & F_z & F_{zz} & F_{zzz} \\ F_x & F_{zx} & F_{zzx} & F_{zzzx} \\ F_y & F_{zy} & F_{zzy} & F_{zzzy} \\ F_{xx} & F_{zxx} & F_{zzxx} & F_{zzzxx} \\ F_{xy} & F_{zxy} & F_{zzxy} & F_{zzzxy} \\ F_{yy} & F_{zyy} & F_{zzyy} & F_{zzzyy} \\ F_{xxx} & F_{zxxx} & F_{zzxxx} & F_{zzzxxx} \\ F_{xyy} & F_{zxyy} & F_{zzxyy} & F_{zzzxyy} \\ F_{xxy} & F_{zxxy} & F_{zzxxy} & F_{zzzxxy} \\ F_{yyy} & F_{zyyy} & F_{zzyyy} & F_{zzzyyy} \end{bmatrix}$$

(The first row is F, F_z, F_{zz}, F_{zzz}. Subsequent rows are all possible derivatives of the first row with respect to x and y, up to order 3.)

You can fill out the entries of the matrix by differentiating F. You can determine its rank by putting the matrix in row-echelon form and counting the number of pivot rows. If the rank is four, then F can't be nomogrammed directly. If the rank is less than four, the test is inconclusive.

Kellogg's second necessary criterion. If F can be represented directly as a nomogram, then you can decompose it as:

$$F(x, y, z) = A_1(x, y)f_1(z) + A_2(x, y)f_2(z) + A_3(x, y)f_3(z) \tag{1}$$

For any* such decomposition, the following two matrices must have zero determinant:

$$\det \begin{bmatrix} A_1 & A_2 & A_3 \\ A_{1x} & A_{2x} & A_{3x} \\ A_{1xx} & A_{2xx} & A_{3xx} \end{bmatrix} = 0$$

$$\det \begin{bmatrix} A_1 & A_2 & A_3 \\ A_{1y} & A_{2y} & A_{3y} \\ A_{1yy} & A_{2yy} & A_{3yy} \end{bmatrix} = 0$$

To apply this test, you must, by whatever means, decompose F into the form (1) above[1]. That decomposition yields functions A_1, A_2, A_3 which you can plug into these matrices. If either matrix has a nonzero determinant, then it is impossible to directly nomograph F. In particular, no other decomposition will work.

[1]See Warmus's algorithm, in Section 3, to see how you can decompose a function like this if you can find places where functions are non-zero.

2 Proof of Kellogg's criteria

I'll describe the flow of the argument, then provide some proofs. Let $F(x, y, z)$ be a function we want to nomogram, and without loss of generality choose one of the variables, say z.

1. First we establish that if we can construct a direct nomogram for F, then[2] the functions F, F_z, F_{zz}, F_{zzz} must be linearly dependent in a certain way. Specifically, we can find coefficient functions c_0, c_1, c_2, c_3, not all zero, such that:

   $$c_0(z)F(x,y,z)+c_1(z)F_z(x,y,z)+c_2(z)F_{zz}(x,y,z)+c_3(z)F_{zzz}(x,y,z) = 0$$

2. Second, we prove a lemma about linear dependence: four bivariate functions $p(x, y), q(x, y), r(x, y), s(x, y)$ are linearly independent if and only if the following matrix has rank less than four:

   $$\begin{bmatrix} p & q & r & s \\ p_x & q_x & r_x & s_x \\ p_y & q_y & r_y & s_y \\ p_{xx} & q_{xx} & r_{xx} & s_{xx} \\ p_{xy} & q_{xy} & r_{xy} & s_{xy} \\ p_{yy} & q_{yy} & r_{yy} & s_{yy} \\ p_{xxx} & q_{xxx} & r_{xxx} & s_{xxx} \\ p_{xyy} & q_{xyy} & r_{xyy} & s_{xyy} \\ p_{xxy} & q_{xxy} & r_{xxy} & s_{xxy} \\ p_{yyy} & q_{yyy} & r_{yyy} & s_{yyy} \end{bmatrix}$$

3. Finally, if you pick a fixed but arbitrary value of z, the functions F, F_z, F_{zz}, F_{zzz} are bivariate functions of x and y. Applying the lemma to these four functions establishes Kellogg's first criterion.

[2] Under typically well-behaved conditions

4. For the second criterion, suppose we can find a direct nomogram for $F(x, y, z)$. By definition, this means we can find nine single-variable functions such that:

$$F(x, y, z) = \det \begin{bmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(y) & g_2(y) & g_3(y) \\ h_1(z) & h_2(z) & h_3(z) \end{bmatrix}$$

.

5. Expanding this determinant along the bottom row, we find an expansion of the form:

$$F(x, y, z) = A_1(x, y)h_1(z) + A_2(x, y)h_2(z) + A_3(x, y)h_3(z)$$

6. There is a result in linear algebra that a matrix with two matching rows must have a zero determinant. For this reason, we know that the following matrices have zero determinant:

$$\det \begin{bmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(y) & g_2(y) & g_3(y) \\ g_1(y) & g_2(y) & g_3(y) \end{bmatrix} = 0$$

$$\det \begin{bmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(y) & g_2(y) & g_3(y) \\ f_1(x) & f_2(x) & f_3(x) \end{bmatrix} = 0$$

7. By expanding these two determinants along the bottom row, we find that:

$$A_1(x, y)g_1(y) + A_2(x, y)g_2(y) + A_3(x, y)g_3(y) = 0 \quad (2)$$

$$A_1(x, y)f_1(x) + A_2(x, y)f_2(x) + A_3(x, y)f_3(x) = 0 \quad (3)$$

where A_1, A_2, A_3 are the same functions as before.

8. The first equation establishes that[3], for any fixed y, the functions A_1, A_2, A_3 are linearly-dependent functions of x. The second equation establishes that, for any fixed x, the functions A_1, A_2, A_3 are linearly-dependent functions of y.

[3] As long as the coefficients are not degenerate

9. We need another lemma about linear independence, this time for univariate functions: Functions $p(x), q(x), r(x)$ are linearly dependent if and only if the following matrix has zero determinant:

$$\det \begin{bmatrix} p & q & r \\ p_x & q_x & r_x \\ p_{xx} & q_{xx} & r_{xx} \end{bmatrix} = 0.$$

10. Applying this lemma to our two equations establishes Kellogg's second criterion.

1 Theorem If you can find a direct nomogram for $F(x, y, z)$, then F, F_z, F_{zz}, F_{zzz} are linearly dependent (as functions of x and y).

Proof. 1. Suppose we can find a direct nomogram for $F(x, y, z)$. By definition, this means we can find nine single-variable functions such that:

$$F(x, y, z) = \det \begin{bmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(y) & g_2(y) & g_3(y) \\ h_1(z) & h_2(z) & h_3(z) \end{bmatrix}.$$

2. If you expand this determinant along a single row, e.g. the final row, you get a decomposition which has the form:

$$F(x, y, z) = A_1(x, y)h_1(z) + A_2(x, y)h_2(z) + A_3(x, y)h_3(z). \tag{4}$$

   This is a weighted sum of functions of z where the coefficients depend only on x and y.

3. If you differentiate this equation (4) three times with respect to z, you obtain a system of four equations:

$$\begin{aligned} F &= A_1(x, y)h_1(z) + A_2(x, y)h_2(z) + A_3(x, y)h_3(z) \\ F_z &= A_1 h_1'(z) + A_2 h_2'(z) + A_3 h_3'(z) \\ F_{zz} &= A_1 h_1''(z) + A_2 h_2''(z) + A_3 h_3''(z) \\ F_{zz} &= A_1 h_1''(z) + A_2 h_2''(z) + A_3 h_3''(z) \\ F_{zzz} &= A_1 h_1'''(z) + A_2 h_2'''(z) + A_3 h_3'''(z) \end{aligned}$$

   Expressed in vector notation, this system is:

$$-\begin{bmatrix} F \\ F_z \\ F_{zz} \\ F_{zzz} \end{bmatrix} + A_1(x, y)\begin{bmatrix} h_1 \\ h_1' \\ h_1'' \\ h_1''' \end{bmatrix} + A_2(x, y)\begin{bmatrix} h_2 \\ h_2' \\ h_2'' \\ h_2''' \end{bmatrix} + A_3(x, y)\begin{bmatrix} h_3 \\ h_3' \\ h_3'' \\ h_3''' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

4. We've found a weighted sum of column vectors which is zero everywhere. If we fix a particular value of x and y, then the A_i become constants and we find that these column vectors are linearly dependent functions of z[4].

   There is a theorem in linear algebra that a matrix has zero determinant if and only if its columns are linearly dependent. So we can assemble these column vectors into a matrix whose determinant is guaranteed to be zero everywhere (i.e. for all $x, y, z,$):

$$\det \begin{bmatrix} F & h_1 & h_2 & h_3 \\ F_z & h_1' & h_2' & h_3' \\ F_{zz} & h_1'' & h_2'' & h_3'' \\ F_{zzz} & h_1''' & h_2''' & h_3''' \end{bmatrix} = 0 \tag{5}$$

5. If you expand the determinant 5 along the first column, you obtain a weighted sum of F, F_z, F_{zz}, F_{zzz}:

$$\begin{vmatrix} h_1 & h_2 & h_3 \\ h_1' & h_2' & h_3' \\ h_1'' & h_2'' & h_3'' \end{vmatrix} F_{zzz} - \begin{vmatrix} h_1 & h_2 & h_3 \\ h_1' & h_2' & h_3' \\ h_1''' & h_2''' & h_3''' \end{vmatrix} F_{zz} + \begin{vmatrix} h_1 & h_2 & h_3 \\ h_1'' & h_2'' & h_3'' \\ h_1''' & h_2''' & h_3''' \end{vmatrix} F_z - \begin{vmatrix} h_1' & h_2' & h_3' \\ h_1'' & h_2'' & h_3'' \\ h_1''' & h_2''' & h_3''' \end{vmatrix} F = 0 \tag{6}$$

   And this equation holds for any choice of x, y, z. Note that F, F_z, F_{zz}, F_{zzz} are functions of (x, y, z), while the coefficients are functions of z only.

6. If we fix a value of z, equation 6 shows[5] that $\{F, F_z, F_{zz}, F_{zzz}\}$ are linearly independent, considered as functions of x and y.

□

2 Lemma A set of four bivariate functions

$$\{p(x, y), q(x, y), r(x, y), s(x, y)\}$$

is linearly dependent if and only if the following 4×10 matrix has rank less than four:

[4] As long as some of the A_i are nonzero.
[5] As long as the coefficients are not trivial.

$$\begin{bmatrix} p & q & r & s \\ p_x & q_x & r_x & s_x \\ p_y & q_y & r_y & s_y \\ p_{xx} & q_{xx} & r_{xx} & s_{xx} \\ p_{xy} & q_{xy} & r_{xy} & s_{xy} \\ p_{yy} & q_{yy} & r_{yy} & s_{yy} \\ p_{xxx} & q_{xxx} & r_{xxx} & s_{xxx} \\ p_{xyy} & q_{xyy} & r_{xyy} & s_{xyy} \\ p_{xxy} & q_{xxy} & r_{xxy} & s_{xxy} \\ p_{yyy} & q_{yyy} & r_{yyy} & s_{yyy} \end{bmatrix}$$

Proof. Here, the matrix consists of the four functions, as well as all of their partial derivatives up to order three. The proof of the result is tedious. See Epstein's *Nomography*, Chapter 8 for details. □

3 Decompositions are well-behaved

Constructing a nomogram for a function $F(x, y, z)$ involves looking for a way to decompose it into a sum of products of single-variable functions like this:

$$F(x, y, z) = \sum_i f_i(x) g_i(y) h_i(z), \tag{7}$$

because this is the form a nomographable function takes when you write it as a determinant:

$$F(x, y, z) = \det \begin{bmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{bmatrix}$$

I found the trial-and-error process uncertain: Sometimes you can't find a decomposition—but maybe you're overlooking one, or maybe one doesn't exist at all. Sometimes you find a decomposition with too many terms—maybe you can simplify it, or maybe there's a completely different-looking decomposition with fewer terms.

Based on the approach of Warmus and my own proofs, I've managed to clear the air on these decompositions. It turns out that decompositions are well-behaved, and there are procedural ways to find them (or prove that they don't exist). In particular:

1. *You can automatically find a minimal decomposition.* There is an algorithm for automatically decomposing a function $f(x, y, z)$ into the form 7. The resulting form is minimal; it can't be simplified further by consolidating terms.

2. *You can automatically simplify decompositions.* Suppose you come up with a decomposition yourself. You can use "linear independence" tests to check whether the decomposition can be simplified, and to produce that simpler decomposition if so.

3. *All irreducible decompositions have the same number of terms.* Here's the scenario I worried about: In a typical case, I might find a decomposition that was had too many terms to be nomographable, check that it was irreducible (i.e. couldn't be made shorter by consolidating terms), and conclude that the function overall couldn't be nomographed. But what if the problem was only a bad

> choice of decomposition? What if it were possible to find a shorter decomposition using completely different factors?
>
> There might exist, for example, a 5-term decomposition and a 3-term decomposition, both of which couldn't be simplified further (they might use different, incommensurate factors such that there'd be no way to simplify the 5-term decomposition into the 3-term one.) In other words, I was worried that I might find an irreducible decomposition that was nonetheless not as short as the shortest possible decomposition.
>
> It turns out this can never happen—all minimal decompositions have the same number of terms. And if you have one minimal decomposition, you can make all others through linear combinations of the factors. This also means that if you use the procedure for finding a minimal decomposition of $f(x, y, z)$ and it has too many terms to be nomographable, you can be sure that no other decomposition will work.

I describe Warmus's procedure for automatically finding a minimal decomposition of any function in the next section. For proofs and more detail, you can consult Warmus's 1959 paper *Nomographic functions*.

The simplification procedure comes from Warmus's work. The theorem is that a decomposition like 7 is minimal if and only if the f_i are all linearly independent, as are the g_i and the h_i. So to simplify, check whether the f_i are all linearly independent, as are the g_i and the h_i. If they aren't, you can consolidate terms.

To check whether terms are linearly independent, you can use the theorem that $f_1, \ldots, f_n$ are linearly independent if and only if you can find $x_1, \ldots, x_n$ such that the matrix $[f_i(x_j)]_{i,j}$ has a nonzero determinant. Practically speaking, choosing random x_i should work.

The proof that every minimal decomposition has the same number of terms comes from this matrix determinant-based definition of linear independence. Proof is here on Math StackExchange, though I intend to write it up myself here.

4 Warmus's constructive procedure

Warmus establishes an automatic procedure for building a nomogram out of any function, or else proving that the function is not nomographable.

Subroutines Warmus's procedure can be done automatically by computer, and depends only on a few simple subroutines:

- *Find support.* Given a function of up to three variables, find a point where that function is nonzero. (Although this is difficult to do deterministically, for practical smooth functions, I expect even random search would quickly find such a point.)
- *Compute determinants.* Compute the determinant of a 2×2 or larger matrix.
- *Find independence certificate.* Given functions $f_1, \ldots, f_n$, find points $x_1, \ldots, x_n$ such that the determinant of the matrix $[f_i(x_j)]_{i,j}$ is nonzero. This is possible if and only if the f_i are linearly independent[6].

Rank The key concept for Warmus's method is the *rank* of a function. A bivariate function $G(u, v)$ has rank n if there exist univariate functions $U_1(u), \ldots, U_n(u)$ and $V_1(v), \ldots, V_n(v)$ such that $G(u, v) = U_1(u)V_1(v) + \ldots + U_n(u)V_n(v)$, and no smaller set of functions will work.

Note that in a decomposition $G \equiv \sum_{i=1}^{n} U_i V_i$, the U_i must all be linearly independent and the V_i must all be linearly independent; otherwise, you could consolidate some terms and form a shorter sum, contradicting the fact that the rank n is minimal.

As a theorem, a function has rank greater than n if and only if there exist $n + 1$ pairs of points $\langle u_i, v_i \rangle$ such that the matrix $[G(u_i, v_j)]$ has nonzero determinant.

Finding the rank and decomposition of a function To find the rank and decomposition of a function G, we'll define a particular sequence of functions $G_0, G_1, G_2, G_3, \ldots$ in terms of G.

[6] And finding an independence certificate is really just finding the support of a particular determinant.

As a base case, define $G_0(x, y) \equiv G$. By induction on k, if G_k is identically zero, then the rank of G is k and we are done. Otherwise, we find a point $\langle a_k, b_k\rangle$ in the support of G_k and define

$$G_{k+1}(x, y) \equiv \frac{1}{G_k(a_k, b_k)} \begin{vmatrix} G_k(a_k, b_k) & G_k(a_k, y) \\ G_k(x, b_k) & G_k(x, y) \end{vmatrix}$$

Generally, this sequence will lead[7] to a function G_n which is identically zero; the value of n is the rank of G. When you've found the rank n of G, you can compute a decomposition $G \equiv \sum_i U_i(u)V_i(v)$ as follows. For $i = 1, \ldots, n$, define

$$U_i(x) \equiv G_{i-1}(x, b_{i-1})$$
$$V_i(y) \equiv \frac{G_{i-1}(a_{i-1}, y)}{G_{i-1}(a_{i-1}, b_{i-1})}$$

And we have, reportedly, that $G = \sum_{i=1}^{n} U_i(x)V_i(y)$.

Rank of functions with three or more arguments You can extend the definition of rank to functions of three or more arguments, basically by dividing the arguments into two nonempty groups so that you have a pair of "arguments" as in the base two case. Divide the function's arguments into two nonempty groups $\vec{x}$ and $\vec{y}$, then as usual define $G_0(\vec{x}, \vec{y}) \equiv G$ and define

$$G_{k+1}(\vec{x}, \vec{y}) \equiv \frac{1}{G_k(\vec{a}_k, \vec{b}_k)} \begin{vmatrix} G_k(\vec{a}_k, \vec{b}_k) & G_k(\vec{a}_k, y) \\ G_k(x, \vec{b}_k) & G_k(\vec{x}, \vec{y}) \end{vmatrix}$$

When you have more than two arguments, there are multiple ways to divide arguments into groups and each way yields a potentially different set of functions and ranks; so when there are more than two arguments, we must be specific and refer to *rank with respect to a particular division $\vec{x}$.*

As a specific case, if $G(x, y, z)$ has three arguments, we can refer to its *rank with respect to x*, which we would compute using terms like:

$$G_{k+1}(x, y, z) \equiv \frac{1}{G_k(a, b, c)} \begin{vmatrix} G_k(a, b, c) & G_k(a, y, z) \\ G_k(x, b, c) & G_k(x, y, z) \end{vmatrix}$$

[7] Some functions might not have finite rank, in which case this process never terminates.

Procedure for constructing nomograms Given a function $F(x, y, z)$:

1. Compute the rank of F with respect to x, y, and z. (Call them r_x, r_y, r_z)

2. Each rank must be two or three; otherwise, the function F is not nomographic.

3. Assume, by rearranging arguments if necessary, that the ranks of x, y, z are in increasing order, so: (2,2,2), (2,2,3), (2,3,3) or (3,3,3). $r_x \leq r_y \leq r_z$.

4. Form a rank decomposition of F with respect to x:

   $$F(x, y, z) \equiv \sum_k X_k(x) G_k(y, z)$$

   This sum will have two or three terms in it, because the rank of F with respect to x is two or three.

5. Compute the rank of each G_k. Each rank must be one or two. If any rank is bigger than two, the function F is not nomographic.

6. If some of the G_k have rank one, there might be a problem. Consult the ranks of F with respect to x, y, and z in order: $r_x \leq r_y \leq r_z$.

   If $r_y = 3$, the function F is not nomographic.

   Otherwise, if *both* of the G_k have rank one, the function is not nomographic.

7. Assume, by rearranging the terms in the sum $F(x, y, z) \equiv \sum_k X_k(x) G_k(y, z)$ if necessary, that the ranks of the G_k are in *decreasing* order from largest to smallest.

8. Having subdivided $G_k(y, z) = \sum_\ell Y_\ell(y) Z_\ell(z)$, we have now formed a sum that looks like

   $$F(x, y, z) = \sum_k X_k(x) \sum_{\ell_k} Y_{\ell_k}(y) Z_{\ell_k}(z).$$

   This form might at first have too many X, Y, or Z terms—we want the number of independent terms to match the rank of F with respect to x, y, and z respectively. We can consolidate extra terms that are linearly dependent. (...)